Received: *November 28, 2001,* Accepted: *December 08, 2001*

Rational conformal field theories with G_2 holonomy

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Abstract: We study conformal field theories for strings propagating on compact, seven-dimensional manifolds with G_2 holonomy. In particular, we describe the construction of rational examples of such models. We argue that analogues of Gepner models are to be constructed based not on $\mathcal{N}=1$ minimal models, but on $\mathbb{Z}_2$ orbifolds of $\mathcal{N}=2$ models. In $\mathbb{Z}_2$ orbifolds of Gepner models times a circle, it turns out that unless all levels are even, there are no new Ramond ground states from twisted sectors. In examples such as the quintic Calabi-Yau, this reflects the fact that the classical geometric orbifold singularity can not be resolved without violating G_2 holonomy. We also comment on supersymmetric boundary states in such theories, which correspond to D-branes wrapping supersymmetric cycles in the geometry.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory.

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1. Introduction

Supersymmetric compactifications of higher-dimensional theories require the internal part of spacetime to have special properties. In the context of supergravity, string, and M-theory, the classical starting points are manifolds of special holonomy, such as Calabi-Yau manifolds, which are $2n$-dimensional manifolds with $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$ holonomy. Additional insight can be gained by probing the geometry with the various extended objects provided by string/M-theory. The goal of the present paper is to explore the geometry of compact seven-dimensional Riemannian manifolds with exceptional holonomy group G_2, using strings as probes. Such manifolds are required for minimal supersymmetry when compactifying $7 = 11 - 4 = 10 - 3$ dimensions.

The subject of string theory on manifolds with G_2 holonomy was started in [1], where the superconformal algebra that is expected to characterize G_2 holonomy was derived. It turns out that the rôle played by the rational U(1) current algebra in the Calabi-Yau context is here played by the tri-critical Ising model.

Let us briefly review how this comes about. For a generic simply connected d-dimensional Riemannian manifold M, the holonomy group is the Lie group $\mathrm{SO}(d)$. Also, the worldsheet fermions of a supersymmetric σ-model on M come in a representation of (local) $\mathrm{SO}(d)$. If M is flat and the holonomy trivial, the fermions are free and give rise to an $\mathfrak{so}(d)_1$ current algebra on the worldsheet. If the holonomy is a non-trivial special subgroup $G \subset \mathrm{SO}(d)$, the worldsheet fermions interact through the coupling to the connection on TM which takes values in the Lie algebra $\mathfrak{g}$ of G. To see what CFT might describe these fermions, recall that gauged WZW models provide a lagrangian description of coset models of conformal field theories. By analogy, the residual symmetry associated with special holonomy G is expected to be the symmetry algebra of the coset CFT,

$$\frac{\mathfrak{so}(d)_1}{\mathfrak{g}} . \tag{1.1}$$

For instance, for $d = 2n$ even, the special holonomy $\mathrm{SU}(n)$ leads to the coset CFT $\mathfrak{so}(2n)_1/\mathfrak{su}(n)$, which is nothing but a $\mathfrak{u}(1)$ current algebra extended by a spin $2n = 2\hat{c}$ field related to the spectral flow operator. If $d = 7$, the exceptional holonomy group $G_2 \subset \mathrm{SO}(7)$ leads to the coset $\mathfrak{so}(7)_1/G_2$, which turns out to have central charge $7/10$ and hence is the CFT of the Ising model at the tri-critical point [1].

Shatashvili and Vafa [1] also derive the extension of the $\mathcal{N} = 1$ superconformal algebra by the tri-critical Ising algebra from a free field representation. In this approach, the spin $3/2$ field of the tri-critical Ising model arises from the closed three-form ϕ that determines the G_2 structure. The stress tensor at central charge $7/10$ is essentially the dual, closed 4-form $*\phi$.

In [2], the superconformal algebra associated with G_2 holonomy was rederived by representing it as the fixed point algebra of the Calabi-Yau three-fold algebra times $\mathfrak{u}(1)$ under the $\mathbb{Z}_2$ automorphism which acts as the mirror automorphism on the $\mathcal{N} = 2$ algebra and as the usual $\mathbb{Z}_2$ automorphism on $\mathfrak{u}(1)$. This mimics the general proposal of Joyce for the construction of G_2 holonomy manifolds, which involves orbifolding a Calabi-Yau manifold times a circle by a $\mathbb{Z}_2$ which acts as an anti-holomorphic involution on the Calabi-Yau and inversion on the circle.

Abstracting the algebra from [1, 2], it becomes a natural problem to look for explicit realizations. Recent work on the subject includes the study of the representation theory of closely related $\mathcal{W}$-algebras [3], the relation between the algebras associated with various special holonomies [4], as well as the construction of modular invariant partition functions for strings on non-compact manifolds with G_2 holonomy [5].

One of the purposes of the present work is to start the construction of explicit examples of compact, rational, conformal field theories with G_2 holonomy. As is well-appreciated in the context of Calabi-Yau manifolds, the study of conformal field theories and their chiral algebras is only one end of the spectrum of approaches to exploring (even perturbative) string theory. A substantial amount of information can already be gained from classical geometry, or from the low-energy space-time theory, as well as from effective worldsheet descriptions such as Landau-Ginzburg models or gauged linear σ-models. How much of this will be available for manifolds of G_2 holonomy remains to be seen. However, exactly solvable models provide structural information and are certainly a welcome starting point, for instance for a better understanding of mirror symmetry for G_2 holonomy manifolds [1].

The most readily accessible class of examples are $\mathbb{Z}_2$ orbifolds of Gepner models times a free boson and fermion. The conformal field theory analysis in section 3 of the present paper reveals that the twisted sector generically does not contribute any new Ramond

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ground states. The only exception are the cases where all levels of the $\mathcal{N} = 2$ minimal models are even. In the geometric description, the corresponding orbifolds of Calabi-Yaus times a circle are singular manifolds with G_2 holonomy. It appears that the orbifold can be smoothened to a G_2 holonomy manifold only if the first Betti number of the fixed point set on the Calabi-Yau is non zero. This is discussed, for instance, in ref. [6]. For the quintic Calabi-Yau, we will verify that this agrees with the conformal field theoretic prediction of absence of twisted sector ground states.

An obvious question one should ask is how generic is the structure unveiled in Gepner models for G_2 holonomy manifolds of physical interest, and whether one can learn anything interesting about M-theory compactifications? Our results, together with the failure of attempts to construct simpler models based, e.g. on tensor products of $\mathcal{N} = 1$ minimal models, seem to indicate that either the best rational models of G_2 holonomy are yet to be found, or that smooth G_2 holonomy manifolds generically do not have a Gepner point. If, on the other hand, the structure found in $\mathbb{Z}_2$ orbifolds of Gepner models turns out to be generic, it should imply that many of the tools used to study Calabi-Yaus manifolds carry over, albeit with severe technical complications, to G_2 holonomy.

Note added. The results discussed in this paper were first presented in [32]. While writing up this note, the preprint [33] was received on the archive. In independent work, the authors of [33] analyze three examples of $\mathbb{Z}_2$ orbifolds of Gepner models. Their conformal field theory results are in agreement with our general formulae.

2. G_2 holonomy CFT

The goal of this section is to develop some general ideas about the construction of RCFTs with G_2 holonomy, guided by the success of Gepner's construction of models which are exact solutions of σ-models in terms of Calabi-Yau manifolds.

2.1 Rational conformal field theories with special holonomy

It is well known that given any rational conformal field with $\mathcal{N} = 2$ supersymmetry and total central charge $\hat{c} = c/3$ integer, one can obtain the internal sector of a supersymmetric string compactification to $D = 10 - 2\hat{c}$ dimensions by projecting the theory onto integral U(1) charge. Namely, integrality of the U(1) charge is equivalent to locality of the chiral spectral flow operator, which upon GSO projection yields a supersymmetry in spacetime. The criterion for consistency of the projection is here modular invariance of the torus partition function.

The most popular class of examples are Gepner's models [7], in which the starting point is a tensor product of $\mathcal{N} = 2$ minimal models, which can be thought of as cosets $\mathrm{SU}(2)_k \times \mathrm{U}(1)/\,\mathrm{U}(1)$. It turns out that Gepner models are related to the exact solution of σ-models on Calabi-Yau manifolds describable as complete intersections in products of projective spaces (see [8, 9] for a summary). In principle, this construction works for Calabi-Yau manifolds of any dimension, although from a physical point of view, $n = \hat{c} = 3$ is of course the most interesting one.

Given the success of Gepner models in describing manifolds of special holonomy $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$, it is natural to wonder whether there exist similar constructions also for the exceptional holonomy groups of Riemannian manifolds, i.e. G_2 in seven dimensions, and Spin(7) in eight dimensions.

2.2 Tensor products of $\mathcal{N} = 1$ minimal models?

For manifolds with G_2 holonomy, there should only be $\mathcal{N} = 1$ supersymmetry on the worldsheet. One might therefore be tempted to generalize Gepner's construction by replacing $\mathcal{N} = 2$ minimal models with $\mathcal{N} = 1$ superconformal minimal models, and to look for a modular invariant that projects out all unwanted states and contains the tri-critical Ising CFT in its maximally extended chiral algebra. It is easy to see that this construction, if it works at all, must be very special.

First of all, it is well known that from the series of $\mathcal{N} = 1$ minimal models, which are the cosets $\mathrm{SU}(2)_k \times \mathrm{SU}(2)_2 / \mathrm{SU}(2)_{k+2}$, only those with even k possess a Ramond ground state, while those with odd k break supersymmetry spontaneously. Thus, in the list of tensor products of $\mathcal{N} = 1$ minimal models with total central charge $c = \sum c_i = \sum (3/2 - 12/(k_i+2)(k_i+4)) = 21/2$, one has to restrict to those were all k_i are even.[1] Then, the only candidate for the space-time supercharge (the analog of the spectral flow operator), with tri-critical Ising dimension 7/16, is built out of the product of Ramond ground states in the individual minimal models. This field however, with minimal model labels $(k/2, k/2+1, 1)$, has the fusion rules

$$\left(\frac{k}{2}, \frac{k}{2}+1, 1\right) \star \left(\frac{k}{2}, \frac{k}{2}+1, 1\right) = \sum_{\substack{l,m,s \text{ even} \\ l/2+m/2+s/2 \text{ even}}} (l, m, s)\,. \tag{2.1}$$

Thus, to reproduce the tri-critical Ising fusion rules, $[7/16] \star [7/16] = [0] + [3/2]$, in the projected tensor product the modular invariant has to be such that most of the terms on the r.h.s. of eq. (2.1) are projected out. In particular, this is not possible with an ordinary simple-current modular invariant. The modular invariant has to be *exceptional*. In terms of modular data, one is looking for a modular invariant in tensor products of SU(2) WZW models with many factors. While a number of exceptional modular invariants for these theories are known [11], none of them appears to have the desired properties. A brief explicit search in the list of candidate tensor products has not revealed any magic in the modular transformations of the products of minimal model characters.

2.3 Orbifolds of $\mathcal{N} = 2$ models

Another possibility for the construction of G_2 holonomy RCFTs is to combine ordinary Gepner models with the results of ref. [2]. Namely, according to [2], any conformal field theory constructed as the $\mathbb{Z}_2$ orbifold theory of a Calabi-Yau model times a free boson and

[1] In particular, this excludes the tri-critical Ising model as an elementary building block.

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fermion has G_2 holonomy, provided the $\mathbb{Z}_2$ acts as

$$\omega: \quad \begin{array}{rcl} T_{CY} & \longmapsto & T_{CY} \\ G_{CY} & \longmapsto & G_{CY} \\ \mathrm{i}\partial X := J_{CY} & \longmapsto & -J_{CY} \\ e^{i\sqrt{\hat{c}}X} & \longmapsto & e^{-i\sqrt{\hat{c}}X} \end{array} \qquad \begin{array}{rcl} T_{S^1} & \longmapsto & T_{S^1} \\ j_{S^1} & \longmapsto & -j_{S^1} \\ \psi_{S^1} & \longmapsto & -\psi_{S^1} \end{array} \tag{2.2}$$

on the symmetry generators of the $CY \times S^1$. In particular, one may model the Calabi-Yau by a Gepner model. We are then interested in the orbifold

$$\frac{\left(\mathrm{Gep} \times S^1\right)}{\mathbb{Z}_2}.$$

2.4 Orbifolds and Extensions

To construct such an orbifold, it is helpful to recall the following well-known fact: the orbifold of any given theory allows for an inverse operation which returns the original theory. In the geometrical context, if the orbifold group is abelian, the inverse operation is often another orbifold by the same group. In the language of conformal field theory, were orbifolding amounts to reducing the chiral symmetry, one recovers the original theory by extending the orbifold theory by all symmetry generators that were broken in the orbifold construction by the introduction of twist fields. In particular, to any $\mathbb{Z}_2$ orbifold of a given CFT $\mathcal{C}$, there is a $\mathbb{Z}_2$ simple-current extension of $\mathcal{C}/\mathbb{Z}_2$, such that $(\mathcal{C}/\mathbb{Z}_2)^{\mathbb{Z}_2} = \mathcal{C}$. We refer to ref. [21] for very explicit and extremely useful formulae on general $\mathbb{Z}_2$ orbifolds, and WZW orbifolds in particular. To construct G_2 holonomy models starting from Gepner models, we thus use the formula

$$\frac{\left(\mathrm{Gep} \times S^1\right)}{\mathbb{Z}_2} = \left(\frac{\mathrm{Gep}}{\mathbb{Z}_2} \times \frac{S^1}{\mathbb{Z}_2}\right)^{\mathbb{Z}_2}, \tag{2.3}$$

where we extend by the off-diagonal $\mathbb{Z}_2$.

In turn, the $\mathbb{Z}_2$ orbifold of a Gepner model is most easily constructed by thinking of a Gepner model as an extended tensor product of $\mathcal{N} = 2$ minimal models. Namely, one orbifolds the individual models, and reconstructs the orbifold of the Gepner model by appropriately extending their tensor product.

We would like to mention an obvious problem at this point. Before orbifolding, the spectral flow operator which we desire to include in the chiral algebra is a primary field of quantum dimension one, with easy to implement $\mathbb{Z}_H$ selection rules. In fact, simple-current theory [10] provides general solutions for the corresponding CFT problem. The problem is that after orbifolding, the generator of the extending sector is a primary field of quantum dimension 2, and there is no generally known prescription for implementing the extension.

The following simple argument shows why orbifold and extension in fact do commute, at least at the level of modular invariance. We may write the partition function of the orbifold before extension as

$$Z_{\mathrm{orb}} = \frac{1}{2}\left(Z_{++} + Z_{+-} + Z_{-+} + Z_{--}\right),$$

where the terms on the r.h.s. are the partition functions with twisted boundary conditions in space and/or time direction on the torus worldsheet. Since Z_{++} is nothing but the original partition function, it is in particular modular invariant, as is the combination of

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the remaining three terms, $Z_{+-} + Z_{-+} + Z_{--}$. Thus, if we wish to construct the orbifold of an extension, we can consider the modular invariant expression

$$Z_{\rm orb}^{\rm ext} = \frac{1}{2}\left(Z_{++}^{\rm ext} + Z_{+-} + Z_{-+} + Z_{--}\right), \tag{2.4}$$

where $Z_{++}^{\rm ext} \equiv Z^{\rm ext}$ is the partition function of the theory obtained by extension alone. $Z^{\rm ext}$ is modular invariant by construction. To ensure that eq. (2.4) indeed is the desired partition function, it suffices to check that the extension does not modify the structure of twisted sectors. In other words, one has to verify that all symmetric sectors of the theory appear in the extension $Z^{\rm ext}$ and that the simple current does not alter the twining characters. A simple example where this condition is satisfied is when one extends by a simple current of odd order and the orbifold group acts by charge conjugation. But more general results would certainly be welcome. One might expect the new conformal field theory methods which have emerged in ref. [22] to be powerful enough to treat such cases.

2.5 Supersymmetric boundary conditions and Cardy states in the tri-critical Ising model

The CFT of the tri-critical Ising model present in the chiral algebra associated with G_2 holonomy has another important application when it comes to specifying possible supersymmetric boundary conditions on the worldsheet, and is hence relevant for the geometry of D-branes wrapping supersymmetric cycles in the G_2 holonomy manifold. Let us pause here from the main thread of the paper to explain this connection.

When constructing boundary conditions in conformal field theory as a worldsheet description of D-branes in string theory, it is important to understand the realization of symmetries, and in particular, the conditions imposed by worldsheet and space-time supersymmetry [12]–[15]. More precisely, because string theory requires gauging $\mathcal{N} = 1$ superconformal symmetry on the worldsheet, one has to require that the boundary conditions on the worldsheet be superconformal invariant. For the corresponding boundary states, this imposes conditions of the form,

$$\left(L_n - \bar{L}_{-n}\right) | \, a \, \rangle\!\rangle = 0 \, , \qquad \left(G_r + i\,\eta\,\bar{G}_{-r}\right) | \, a \, \rangle\!\rangle = 0 \, . \tag{2.5}$$

These conditions must be satisfied exactly by the boundary states. Any additional operators that might be present in the chiral algebra can, however, also be realized twistedly, i.e.

$$\left(\mathcal{O}_n - (-1)^{\Delta_{\mathcal{O}}}\omega\left(\bar{\mathcal{O}}_{-n}\right)\right) | \, a \, \rangle\!\rangle = 0 \, , \tag{2.6}$$

where ω is some automorphism of the chiral algebra. In particular, fields responsible for spacetime supersymmetry must satisfy a condition of the form (2.6), such that the branes preserve a combination of left- and right-moving spacetime supersymmetry charges, as dictated by ω.

A good example for these considerations arises in Calabi-Yau compactifications. Recall that in this case, the $\mathcal{N} = 1$ superconformal algebra is extended by the chiral algebra of a free boson at a rational radius, $R^2 = \hat{c}$. Conformal boundary conditions for the $\mathfrak{u}(1)$

current $J = i\sqrt{\hat{c}}\,\partial X$ are

$$\begin{aligned} \text{Neumann} \qquad & \big(J_n + \bar{J}_{-n}\big)\,|\,a\,\rangle\!\rangle = 0\,, \\ \text{and Dirichlet} \qquad & \big(J_n - \bar{J}_{-n}\big)\,|\,a\,\rangle\!\rangle = 0\,. \end{aligned}$$

Furthermore, the extension by the spectral flow operator, $e^{i\sqrt{\hat{c}}X}$, allows the detection of the position of the Dirichlet boundary condition or the value of the Wilson line as an automorphism type of the boundary condition, i.e. one has the gluing condition,

$$e^{i\sqrt{\hat{c}}\,X_{\rm L}(z)} = e^{2i\varphi} e^{\pm i\sqrt{\hat{c}}\,X_{\rm R}(\bar{z})}\,,$$

at the worldsheet boundary $z = \bar{z}$. Note that φ is well-defined only up to shifts by π and hence only specifies the boundary condition for X modulo $2\pi/\sqrt{\hat{c}}$ on a circle of radius $\sqrt{\hat{c}}$. This "grade" φ of the boundary condition controls many essential aspects of spacetime supersymmetry for D-branes on Calabi-Yaus, and is very important in applications. This was particularly emphasized in ref. [17], and further studied, e.g. in [16, 17].

Similar reasoning is easily applied for G_2 holonomy, where the rôle of the rational u(1) current algebra is taken over by the tri-critical Ising CFT. It is known from the work of Cardy [18] that symmetry preserving boundary conditions in rational conformal field theories are in one-to-one correspondence to primary fields. In the bosonic description, there are therefore six possible types of symmetric boundary conditions, labelled by the six primary fields of the tri-critical Ising model. In a supersymmetric language, we expect branes and anti-branes as well as two possible spin structures for the fermions on the open string. As regards spacetime supersymmetry, the six primary fields of the tri-critical Ising model are accordingly divided into two groups,

$$\begin{array}{cccc} [0] & \left[\frac{3}{2}\right] & \left[\frac{1}{10}\right] & \left[\frac{3}{5}\right] \\ \multicolumn{2}{c}{\left[\frac{7}{16}\right]} & \multicolumn{2}{c}{\left[\frac{3}{80}\right]\,.} \end{array} \tag{2.7}$$

The doubling on the first line reflects the existence of branes and anti-branes, while the states in the second line are necessary for obtaining an open string Ramond sector. It would be interesting to find an explicit connection between the abstract labelling (2.7) and geometric boundary conditions at large volume, discussed for example in [19].

3. Examples

We now turn to the main results of this paper, which is the construction of examples of rational conformal field theories with G_2 holonomy. The general strategy, as outlined in the previous section, is to use orbifolds of $\mathcal{N} = 2$ (minimal) models as basic building blocks, and to perform a modular invariant projection on their tensor products. At the end of this section, we will give a geometric interpretation to some of our results.

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3.1 Orbifolds of $N = 2$ minimal models

We thus need $\mathbb{Z}_2$ orbifolds of $\mathcal{N} = 2$ minimal models. Note that these orbifolds are not the ones that are usually studied in the context of LG orbifolds, with action $\Phi \mapsto e^{2\pi \mathrm{i}/h}\Phi$ on the Landau–Ginzburg field Φ. In the CFT, the latter orbifolds arise from "dividing out" simple-current symmetries. The orbifolded theories differ from the original ones only by a modification of the modular invariant partition function. In particular, the symmetry algebra of the orbifold models still is the $\mathcal{N} = 2$ super-Virasoro algebra.

The orbifolds of present interest arise from dividing out the mirror automorphism of the $\mathcal{N} = 2$ super-Virasoro algebra,

$$\omega: \quad L_n \longmapsto L_n\,, \qquad G_r^{\pm} \longmapsto G_r^{\mp}\,, \qquad J_n \longmapsto -J_n\,. \tag{3.1}$$

In particular, the orbifold breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$.

The induced action on the primary fields is

$$\omega^*: \quad \phi_{(l,m,s)} \longmapsto \phi_{(l,-m,-s)}\,. \tag{3.2}$$

Here, $0 \le l \le k$, $m \in \mathbb{Z}_{2(k+2)}$, $s \in \mathbb{Z}_4$, with $l + m + s$ even, are the ordinary minimal model labels. Taking into account the field identification $(l,m,s) \equiv (k - l, m + h, s + 2)$, one easily enumerates the fixed points of ω^*: in the NS sector one has $(l,0,0)$, $(l,h,0)$, where $l = 0,\ldots,k$ has the appropriate parity, while in the R sector, one finds $(k/2,h/2,1)$ and $(k/2,h/2,-1)$, if k is even. This gives a total of $k + 1$ fixed fields if k is odd, and $k + 4$ if k is even. By general properties of $\mathbb{Z}_2$ orbifolds [20, 21], each of these fixed fields yields two primary fields in the untwisted sector, and gives rise to two fields in the twisted sector.

To compute the twisted sectors note that the $\mathcal{N} = 2$ minimal models have a representation as cosets, $\mathrm{SU}(2)_k \times \mathrm{U}(1)/\,\mathrm{U}(1)$. Therefore, the orbifold of interest can be thought of as a "coset of orbifolds". We note that this involves a non-standard CFT construction, and we do not have a proof that one may interchange cosetting and orbifolding in general. Here, however, we are dealing with a very simple case. The coset does not have field identification fixed points, and the $\mathbb{Z}_2$ automorphism is simply given by charge conjugation. For the $\mathrm{U}(1)$ factors, the classic reference is [20], while for $\mathrm{SU}(2)$, the relevant results can be extracted from [21]. One may then proceed formally as for the usual coset construction to obtain the "orbifold of the coset" as a "coset of orbifolds".

According to this prescription, the twisted sectors in the orbifold of the $\mathcal{N} = 2$ minimal model are given by combinations $(\lambda,\mu,\sigma,\psi)$. Here, $\lambda = 0,\ldots,k$ characterizes the twisted sector in the $\mathrm{SU}(2)$ part, $\mu,\sigma = 0,1$ distinguish the two twisted sectors of the $\mathrm{U}(1)$ factors, while $\psi = \pm$ is the usual degeneracy label between two twist fields in the same twisted sector. In addition, the coset construction restricts $\lambda + \mu + \sigma$ to be even, and implies the field identification $(\lambda,\mu,\sigma,\psi) \equiv (k - \lambda,\mu + h,\sigma,\psi)$. It is easy to show that $\sigma = 0$ corresponds to fields in the Ramond sector, while those with $\sigma = 1$ are in the NS sector.

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Furthermore, one may compute the conformal dimension in the twisted sectors,

$$\Delta_{(\lambda,\mu,\sigma,\psi)} = \frac{c}{24} + \frac{(k-2\lambda)^2}{16h}\,, \tag{3.3}$$

if $\psi = +$, and one has to add $1/2$ if $\psi = -$. Here $c = 3k/h$ is the central charge of the minimal model, and $h = k+2$.

It is amusing to note that the conformal dimension (3.3) is independent of σ and is thus the same for R and NS sector fields. One may wonder how this is consistent with spacetime supersymmetry. To understand this, let us recall a few facts about the $\mathbb{Z}_2$ orbifold of the free boson/fermion system that represents the S^1 in the theory (2.3). In the twisted sector of (supersymmetric) $S^1/\mathbb{Z}_2$, fermions in the R sector are antiperiodic, while fermions in the NS sector are periodic. On the other hand, the bosons are always antiperiodic. Therefore, the contribution from the S^1 to the conformal dimension is $1/16$ in the R sector and $1/16 + 1/16$ in the NS sector. The difference is just the expected difference between R and NS sector conformal weights for the supersymmetric compactification of 7 dimensions. Something similar happens in 8 dimensions. These considerations also show that there will be no tachyons from the twisted sectors.

Another important consequence of the formula (3.3) is that there are R ground state (which are characterized by $\Delta = c/24$) if and only if k is even and $\lambda = k/2$. In particular, when we combine the (orbifolds of) $\mathcal{N} = 2$ minimal model to form our G_2 holonomy CFT, there can be R ground states in the twisted sector if and only if all levels of the minimal models involved are even.

The quintic model is the simplest example where there actually are no ground states from twisted sectors. This Gepner model consists of 5 minimal models at level 3. It has 208 RR ground states, in one-to-one correspondence with the elements of the cohomology of the quintic. When we multiply with the two ground states of the boson/fermion system, and divide by the $\mathbb{Z}_2$ orbifold action, we are again left with $208 = b_0+b_2+b_3+b_4+b_5+b_7 = 2(1+b_2+b_3)$ ground states.[2] The geometry corresponding to the $\mathbb{Z}_2$ orbifold of the quintic Gepner model times S^1 is thus predicted to have

$$b_2 + b_3 = 103\,. \tag{3.4}$$

3.2 Geometric Interpretation

We now turn to the geometric picture of the construction. We will focus on the quintic Calabi-Yau, but some of our arguments will be more general. The quintic hypersurface in $\mathbb{P}^4$ at the Gepner point is given by

$$Q := \left\{ \sum_{i=1}^{5} \left(Z^i\right)^5 = 0,\ \ \left(Z^1,\ldots,Z^5\right) \in \mathbb{P}^4 \right\}. \tag{3.5}$$

Quite generally, we expect the conformal field theories of the previous subsection to describe geometries of the form $(CY_3 \times S^1)/\mathbb{Z}_2$, where CY_3 is the Calabi-Yau three-fold associated

[2] The fact that $b_1 = 0$ can be justified from a geometric perspective by the fact that the only non-trivial 1-cycle in the initial geometry does not survive the $\mathbb{Z}_2$ projection.

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with the Gepner model [8, 9]. Here, the geometric action of $\mathbb{Z}_2$ is extracted from the CFT construction. More precisely, we will see that the $\mathbb{Z}_2$ acts by an antiholomorphic involution whose fixed point set is a special lagrangian cycle in CY_3. In particular, the G_2 structure on $CY_3 \times S^1$, which according to [30] is specified by the 3-form

$$\phi = \mathrm{Re}(\Omega) + J \wedge \theta \tag{3.6}$$

is left invariant. Here, Ω is the (unique) holomorphic 3-form on the Calabi-Yau three-fold CY_3, J is its Kähler form, while θ is the generator of $H^1(S^1, \mathbb{Z})$.

A candidate geometric $\mathbb{Z}_2$ action on CY_3 is best derived in the context of Landau-Ginzburg models, whose IR fixed points are the minimal model building blocks of the Gepner model. The important observation is that ω acts on the left- and right-moving sectors of the CFT in the same way, namely by the simple sign flip of the U(1) charge, see eq. (3.2). In the effective LG description, this is reproduced by the action $\omega:\ \Phi \mapsto \bar{\Phi}$, which is indeed a symmetry of the LG action. We note that the orbifold amounts to gauging the symmetry between chiral and anti-chiral fields of the theory, and can not be confused with the mirror symmetry of [23], which exchanges chiral with twisted-chiral fields. We conclude that the geometric action of ω on the Calabi-Yau is simply complex conjugation of the coordinates, $Z^i \mapsto \bar{Z}^i$. It is then easy to see that the G_2 structure (3.6) is preserved by ω, since both J and θ change sign while $\mathrm{Re}(\Omega)$ is invariant.

It is worthwile pointing out that the identification of a geometric action of ω need not be unique. For example, we can easily generalize it by twisting with automorphisms of the Calabi-Yau space. For the quintic, this is achieved by including phases [24] that preserve the form of eq. (3.5). More precisely, we can consider the $\mathbb{Z}_2$ action

$$\omega:\ \Phi_i \longmapsto \rho_i \overline{\Phi}_i \qquad \text{with} \qquad \rho_i^5 = 1\,, \qquad \text{and} \qquad \prod \rho_i = 1\,. \tag{3.7}$$

For the given G_2 structure (3.6), there are then 125 choices of quintets $(\rho_1, \ldots, \rho_5)$. At the Gepner point, these choices are all equivalent by coordinate transformations, but not so away from it.

The fixed point loci of these involutions (3.7) are special lagrangian cycles, described by the equations

$$\mathrm{Im}\left((\rho_i)^{-1/2} Z^i\right) = 0\,.$$

Some properties of these cycles can easily be discovered by studying the restriction of the quintic equation to the fixed point set, which reads

$$\sum_{i=1}^{5} \left(\mathrm{Re}\left((\rho_i)^{-1/2} Z^i\right)\right)^5 = 0\,.$$

It is easy to see that this equation has unique real solutions for all choices of $(\rho_1, \ldots, \rho_5)$. Furthermore, since the equation is projective, the fixed point locus is $L = \mathbb{RP}^3$, whose fundamental group is

$$\pi_1\left(\mathbb{RP}^3\right) = \mathbb{Z}_2\,.$$

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Since $\mathbb{RP}^3$ is orientable, Poincaré duality applies and we find that

$$H^1(L, \mathbb{Z}) = 0\,. \tag{3.8}$$

This last observation turns out to be very important in comparing the geometry with the CFT results.

Since the involutions (3.7) that we wish to divide out have fixed points, we must study the structure of the resulting space around those points and see whether the singularities can be resolved. Because the $\mathbb{RP}^3$'s fixed by (3.7) are (special) lagrangian cycles, the local structure of the quintic around them is that of the total space of the cotangent bundle $T^*\mathbb{RP}^3$, and ω acts on the cotangent directions.

The fixed point set of ω in $CY_3 \times S^1$ actually consists of two copies of the aforementioned $\mathbb{RP}^3$ cycles, because of the two fixed points on S^1. Using open sets $\{U_i\}$ to cover the fixed cycles, the local geometry around the fixed point loci is that of the A_1 singularity

$$U_i \times \left(\frac{\mathbb{R}^4}{\mathbb{Z}_2}\right), \tag{3.9}$$

where $\mathbb{Z}_2$ acts on $\mathbb{R}^4$ by reversing the sign of all coordinates, $x_i \mapsto -x_i$, for $i = 1, \ldots, 4$.

Following Joyce [25, 26], over each open set U_i we can resolve the A_1 singularity in the normal bundle while preserving the natural G_2 structure [28, 29] that is invariant under the $\mathbb{Z}_2$ action,

$$\phi = \gamma_1 \wedge \delta_1 + \gamma_2 \wedge \delta_2 + \gamma_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3\,. \tag{3.10}$$

Here, the δ_i are constant orthonormal 1-forms on U_i and γ_i constant 2-forms on the resolved $\mathbb{R}^4/\mathbb{Z}_2$. Thus, it is certainly possible to resolve the singularities. The non-trivial question is whether the local G_2 structures can be glued together to a global one. The operation can be split into two steps: first we reconstruct the geometry of $(T^*\mathbb{RP}^3 \times \mathbb{R})/\mathbb{Z}_2$ around each of the two disjoint components of the fixed point set and then reconstruct the full manifold.

The resolutions of the singularities considered in refs. [25, 26] can be endowed with a global G_2 structure. However, in our cases it is not hard to see that the gluing fails to produce a globally-defined parallel three-form. The fundamental result of [27] implies that we should look for a harmonic three-form. However, since $H^1(\mathbb{RP}^3, \mathbb{R}) = 0$ it follows that δ_i are exact 1-forms with support on $\mathbb{RP}^3$. This in turn implies that ϕ is exact and thus, using Hodge decomposition, cannot be a harmonic form. Thus, we conclude that the manifold resulting from the resolution of the singularities in the normal bundle of $\mathbb{RP}^3$ does not have a globally defined torsion free G_2 structure.

The fact that the singularities cannot be resolved while preserving the G_2 structure (3.6) is the geometric picture of the absence of twisted sector RR ground states in the CFT. We are thus really studying string theory on a singular manifold of G_2 holonomy. But the CFT, which was constructed as an ordinary orbifold, is completely non singular. In particular, we would not expect any additional non-perturbative massless states. This does not exclude the possibility that the CFT can be deformed by exactly marginal operators that do not fall in supersymmetry multiplets and thus completely break spacetime super-

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symmetry. This is of course consistent with the results above since lack of G_2 structure prevents the existence of parallel spinors. It would be interesting to systematically analyze such possibilities.

Due to the presence of singularities, the geometry and topology of this space are somewhat subtle to define. One can count the invariant forms as in [6], but this does not generally yield the correct dimension of the cohomology groups since one has to take into account additional contributions arising from the collapsed cycles. In our case, starting from the cohomology groups of the quintic and of the circle we find that there are no invariant 1-forms since ω acts on S^1 by reversing the sign of the coordinate and there are no invariant 2-forms since the 2-form on the quintic changes sign under the action of ω. The invariant 3-forms one can construct are: the real part of any element of $H^{2,1}(Q)$, the real part of Ω, and $J \wedge \theta$ where J is the Kähler form on the quintic and θ is the 1-form on the circle. We thus find

$$b_0 = 1\,, \qquad b_1 = 0\,, \qquad b_2 = 0\,, \qquad b_3 = 103\,.$$

From a geometric perspective it is not clear whether we should count any collapsed cycles. As discussed in section 3, the CFT prediction for the stringy Betti numbers is

$$b_2^s + b_3^s = 103\,,$$

which suggests that for the $(Q \times S^1)/\mathbb{Z}_2$ the stringy Betti numbers count the $\mathbb{Z}_2$ invariant forms.

4. Conclusions

In this paper, we have presented examples of rational CFTs describing string propagation on manifolds with G_2 holonomy. The chiral algebra of these models was obtained as the fixed point algebra of the $\mathcal{N} = 2$ superconformal algebra under the mirror automorphism, extended by an appropriate spin 3/2 field. Due to the equivalence with $\mathbb{Z}_2$ orbifolds of Gepner models times a circle, the chiral algebra contains the algebra that characterizes G_2 holonomy CFT as a subalgebra.

The construction is interesting for the conformal field theorist just as much as for the geometer. We have mentioned the CFT aspects in detail in the text, so let us here rather comment on some geometric implications.

One of the main results from CFT is that there are typically no twisted sector RR ground states and supersymmetric moduli. We have argued for the quintic that this reflects the fact that the orbifold cannot be resolved to a smooth G_2 holonomy manifold. It is natural to expect other Gepner models to lead to a similar structure. It may well be possible that the models obtained in this fashion are atypical, and do not reflect generic properties of G_2 holonomy manifolds. Nevertheless, for the manifolds that can be constructed in this way, it follows that all deformations of the manifold are actually inherited from the Calabi-Yau, and hence there should be an interesting relation between the respective moduli spaces. This should have interesting physical signatures. In particular, due to the absence of twisted

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massless states, the $\mathcal{N}=2$ three-dimensional field theories obtained by compactifying string theory on such manifolds are just *field theory orbifolds* of $\mathcal{N}=4$ theories. Depending on the particular realization and by analogy with [31], this in turn would imply that, in an appropriately chosen regime, some correlation functions are the same as in the parent theory. It would be interesting to analyze this in detail.

Acknowledgments

We would like to thank Samson Shatashvili for his inspiring lecture on Exceptional Magic at the Les Houches 2001 Summer School. R. R. thanks Martin Roček for getting him interested in G_2 holonomy CFTs. We are grateful to Lilia Anguelova, Harald Ita, and especially Christian Römelsberger for very helpful discussions. The research of R. R. was supported in part by the DOE under Grant No. 91ER40618(3N) and by the National Science Foundation under Grant No. PHY00-98395(6T). The research of J. W. was supported in part by the National Science Foundation under Grant No. PHY99-07949.

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